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TRANSIENT RESPONSE OF A MULTI-SPAN BEAM ON NON-SYMMETRIC PIECEWISE-LINEAR SUPPORTS

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Abstract—The equation of motion in matrix form of a multi-span beam on non-symmetric, piecewise-linear supports is formulated using Hamilton's principle and the assumed mode method. The beam is subject to a stationary unsteady load or a constant moving load. The non-symmetric, piecewise-linear supports are used as models for non-symmetric, non-linear magnetic levitation supports in various machines. Numerical simulations are carried out for some sample problems to demonstrate the differences in transient responses between multi-span beams on linear and piecewise-linear supports.

1. INTRODUCTION

Many studies have been reported for the responses of a multi-span beam acted upon by moving loads in connection with the machining process and vibration behavior of railway tracks and bridges. The model was also pointed out by Katz *et al.* (1987) to be relevant to the ballistic machining system described by Flom *et al.* (1984) and to other possible machining processes where the axial speed of the workpiece is very much higher than its rotational speed. The classical solution of a beam subject to a constant moving load was presented by Timoshenko (1922). Exact solution for the resulting partial differential equation using infinite series was presented by Ayre *et al.* (1950) for the dynamic response of a symmetric two-span beam subject to a moving load. Numerical results using infinite series were also presented in the book by Frýba (1972) for continuous beams with two and six equal spans. Studies by Nelson and Conover (1971), Benedetti (1974), Steele (1967), Florence (1965), and Katz *et al.* (1987) included the effects of elastic foundation, moving mass, and deflection dependent moving loads. A related problem of moving loads on beams resting on tensionless Winkler foundation was analysed by Kenny (1954), Tsai and Westmann (1967), Weitsman (1971), Kameswara Rao (1974), Adams and Bogy (1975), and Choros and Adams (1979). The response of a beam on bilinear foundation subject to an arbitrary static load was presented by Johnson and Kouskoulas (1973).

The free vibration of a multi-span beam has also been studied extensively with respect to finding the natural frequencies and modes of a beam on linear spring supports or a beam with intermediate point constraints. A comprehensive list of references can be found in the work by Bergman and McFarland (1988). The dynamic response of a beam with intermediate point constraints subject to a moving load was presented by Lee (1993) using Hamilton's principle and the assumed mode method. Numerical results were also reported by Maurizi and Bambill de Rossit (1987), Kim and Dickinson (1988) and Kukla (1991).

The present problem for the transient response of a multi-span beam on non-symmetric, non-linear supports was analysed by Nagaya and Kato (1990) using a combination of Laplace Transform, Fourier Series expansion and Residue Theorem. The non-symmetric, non-linear supports are used as models for magnetic levitation supports which can be found frequently in linear vehicle guideways and various machines. The non-linear supports were approximated by piecewise-linear supports. Numerical results were reported for a beam subject to a stationary decaying load. Numerical study for a beam subject to a moving load was not included in their work.

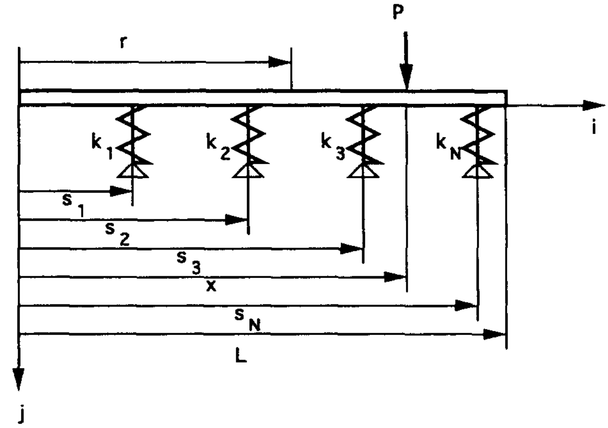

Fig. 1. A multi-span beam on spring supports.

In the present study, the equation of motion in matrix form for a multi-span beam on non-symmetric, piecewise-linear supports subject to a stationary unsteady load or a constant moving load is formulated using Hamilton's principle and the assumed mode method. Numerical simulations are to be carried out for some sample problems to demonstrate the differences in transient responses between multi-span beams on linear and piecewise-linear supports.

2. THEORY AND FORMULATIONS

The beam considered is a uniform beam of length L on a series of spring supports shown in Fig. 1. The two ends of the beam are assumed to be free. The coordinate system, shown in Fig. 1, is assumed to be fixed in an inertial frame with the **i** unit vector parallel to the undeformed beam. A load P located at $r = x$ is applied in the **j** direction. x is fixed for a stationary load and can be expressed as a function of time for a moving load. The assumptions made in the following formulation are that transverse deflections are small so that the dynamic behavior of the beam is governed by the Euler beam theory. Moreover, all the transverse deflections occur on the same plane defined by the **i** and **j** unit vectors.

The characteristic of the spring supports is shown in Fig. 2. The piecewise-linear characteristic is used as an approximation for the non-linear characteristic of a magnetic

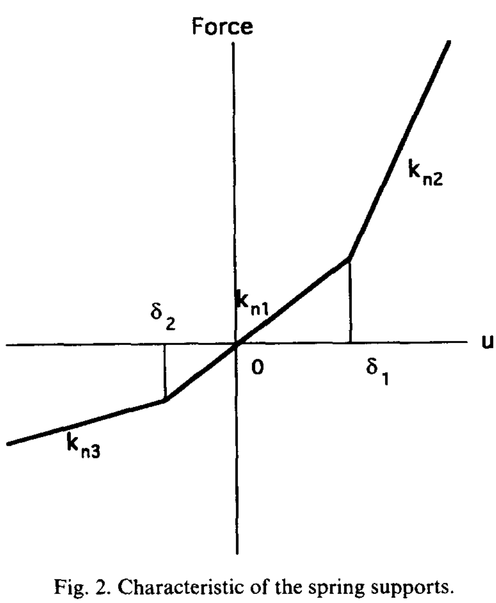

Fig. 2. Characteristic of the spring supports.

levitation support (Nagaya and Kato, 1990). The support has a hard-spring characteristic when the beam moves in the positive **j** direction and a soft-spring characteristic when the beam moves in the other direction. The spring constants for the three linear segments in Fig. 2 are denoted by k_{n1}, k_{n2} and k_{n3}. For simplicity, the piecewise-linear segments of the spring characteristic are chosen such that $|\delta_1|$ is equal to $|\delta_2|$. All the spring supports are assumed to be identical with the same spring characteristic as shown in Fig. 2.

The position vector of a general point p on the deformed beam is given by

$$\mathbf{p} = r\mathbf{i}+u\mathbf{j}. \tag{1}$$

The velocity at the point is

$$\mathbf{v}_p = \dot{u}\mathbf{j}, \tag{2}$$

where $\dot{u} = \mathrm{d}u/\mathrm{d}t$.

The kinetic energy T of the beam is

$$T = \tfrac{1}{2}m\int_0^L \dot{u}^2\,\mathrm{d}r, \tag{3}$$

where m is the mass of the beam per unit length.

Assuming Euler's beam theory, the elastic strain energy of the beam due to bending is

$$V_\varepsilon = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 u}{\partial r^2}\right)^2 \mathrm{d}r, \tag{4}$$

where E and I are the Young's modulus and the principal second area moment of inertia of the cross-section of the beam.

The work done due to the load P is

$$W = Pu(x) \tag{5}$$

where $u(x)$ is the deflection of the beam under the moving load at $r = x$.

The potential energy due to a spring support located at $r = s_p$ can be computed from the area under the curve in Fig. 2. For $-\delta \leqslant u(s_p) \leqslant \delta$, the support behaves as a linear spring of stiffness k_{n1} with the potential energy given by

$$V_{sp} = \tfrac{1}{2}k_{n1}u^2(s_p). \tag{6}$$

For $u(s_p) > \delta$, the area under the curve on the right upper quadrant in Fig. 2 can be taken as the sum of two triangular sections and a rectangular section. The potential energy is given by

$$V_{sp} = \tfrac{1}{2}k_{n1}\delta^2+\tfrac{1}{2}k_{n2}(u(s_p)-\delta)^2+k_{n1}\delta(u(s_p)-\delta). \tag{7}$$

Similarly, for $u(s_p) < \delta$ and bearing in mind that $u(s_p)$ is a negative quantity, the potential energy is given by

$$V_{sp} = \tfrac{1}{2}k_{n1}\delta^2+\tfrac{1}{2}k_{n3}(u(s_p)+\delta)^2-k_{n1}\delta(u(s_p)+\delta). \tag{8}$$

The potential energy due to a spring support could therefore be expressed as

$$V_{sp} = \tfrac{1}{2}k_{sp}(u^2(s_p)+\delta^2)\pm(k_{n1}-k_{sp})\delta u(s_p)-\tfrac{1}{2}k_{n1}\delta^2, \tag{9}$$

where $\delta = |\delta_1| = |\delta_2|$ and

$$k_{sp} = \begin{cases} k_{n2} & \text{for} \quad u(s_p) > \delta, \\ k_{n1} & \text{for} \quad -\delta \leqslant u(s_p) \leqslant \delta, \\ k_{n3} & \text{for} \quad u(s_p) < -\delta. \end{cases} \tag{10}$$

The sign before the term $(k_{n1} - k_{sp})\delta u(s_p)$ is negative for $u(s_p) < -\delta$ and positive otherwise.

The total potential energy due to the spring supports is

$$V_s = \sum_{p=1}^{N} V_{sp}, \tag{11}$$

where N is the number of spring supports. It can be seen from the above expression that increasing the number of spring supports will only increase the number of terms for the addition. It will not directly affect the expressions for kinetic energy and strain energy.

Using the assumed mode method, the quantity u can be expressed as

$$u = \sum_{i=1}^{n} q_i(t)\phi_i(r), \tag{12}$$

where ϕ_i are spatial functions that satisfy the boundary conditions at the two ends of the beam. For the present study, ϕ_i are assumed to be the modal functions for the vibration of a uniform beam free at both ends. ϕ_1 and ϕ_2 are defined respectively as the modal functions for rigid body translation and rigid body rotation.

The assumed form of u enables the kinetic energy, the strain energy, the potential energy due to the spring supports, and the work done by the moving or stationary load P to be expressed in matrix form as follows:

$$T = \tfrac{1}{2} m \dot{\mathbf{q}}^{\mathrm{T}} \mathbf{H} \dot{\mathbf{q}}, \tag{13}$$

$$V_\varepsilon = \tfrac{1}{2} EI \mathbf{q}^{\mathrm{T}} \mathbf{M} \mathbf{q}, \tag{14}$$

$$V_{sp} = \tfrac{1}{2} k_{sp} (\mathbf{q}^{\mathrm{T}} \mathbf{\Phi}_{\mathrm{p}} \mathbf{q} + \mathbf{c}^{\mathrm{T}} \mathbf{\Phi}_{\mathrm{p}} \mathbf{c}) \pm (k_{n1} - k_{sp}) \mathbf{q}^{\mathrm{T}} \mathbf{\Phi}_{\mathrm{p}} \mathbf{c} - \tfrac{1}{2} k_{n1} \mathbf{c}^{\mathrm{T}} \mathbf{\Phi}_{\mathrm{p}} \mathbf{c}, \tag{15}$$

$$W = P \psi^{\mathrm{T}} \mathbf{q}, \tag{16}$$

where $\mathbf{H}$, $\mathbf{M}$ and $\mathbf{\Phi}_p$ are matrices defined as

$$(\mathbf{H})_{ij} = \int_0^L \phi_i \phi_j \, \mathrm{d}r, \tag{17}$$

$$(\mathbf{M})_{ij} = \int_0^L \phi_i'' \phi_j'' \, \mathrm{d}r, \tag{18}$$

$$(\mathbf{\Phi}_{\mathrm{p}})_{ij} = \phi_i(s_p)\phi_j(s_p), \tag{19}$$

where $\phi_i(s_p)$ is the value of ϕ_i evaluated at $r = s_p$. ϕ_i' and ϕ_i'' denote the first and second derivatives of ϕ_i with respect to r. $\mathbf{q}$ and $\dot{\mathbf{q}}$ are $n \times 1$ column vectors consisting of q_i and $\dot{q}_i$ respectively. ψ and $\mathbf{c}$ are vectors defined as

$$(\psi)_i = \phi_i(x), \tag{20}$$

$$(\mathbf{c})_i = \begin{cases} \delta\phi_i & \text{for} \quad i = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

The function ϕ_1 is the modal function for rigid body translation given by $\phi_1(r) = 1$.

All the matrices except ψ are independent of time. ψ needs to be updated as the load is moving over the beam. If the load is stationary, ψ is also independent of time.

The Lagrangian of the beam involving u can be expressed as

$$L = T - V_\varepsilon - V_s + W. \tag{22}$$

The resulting Euler–Lagrange equation is

$$m\mathbf{H}\ddot{\mathbf{q}} + (EI\mathbf{M} + k_{s1}\mathbf{\Phi}_1 + \cdots + k_{sN}\mathbf{\Phi}_N)\mathbf{q} = P\psi \mp (k_{n1} - k_{s1})\mathbf{\Phi}_1\mathbf{c} \mp \cdots \mp (k_{n1} - k_{sN})\mathbf{\Phi}_N\mathbf{c}. \tag{23}$$

For simplicity in the subsequent computations, the following dimensionless quantities are introduced:

$$\tau = t\sqrt{\frac{EI}{mL^4}}, \tag{24}$$

$$\xi = \frac{r}{L}, \tag{25}$$

$$\bar{x} = \frac{x}{L}, \tag{26}$$

$$\bar{\delta} = \frac{\delta}{L}, \tag{27}$$

$$\bar{s}_p = \frac{s_p}{L}, \quad p = 1, 2, \ldots, N, \tag{28}$$

$$\bar{k}_{nj} = \frac{k_{nj}L^3}{EI}, \quad j = 1, 2, 3, \tag{29}$$

$$\bar{k}_{sp} = \frac{k_{sp}L^3}{EI}, \quad p = 1, 2, \ldots, N, \tag{30}$$

$$\bar{P} = \frac{PL^4}{EI}. \tag{31}$$

The dimensionless normalized assumed functions are

$$\phi_1(\xi) = 1, \tag{32}$$

$$\phi_2(\xi) = \sqrt{3}(1 - 2\xi), \tag{33}$$

$$\phi_i(\xi) = \cos\lambda_{i-2}\xi + \cosh\lambda_{i-2}\xi - \gamma_{i-2}(\sin\lambda_{i-2}\xi + \sinh\lambda_{i-2}\xi) \quad (i = 3, \ldots, n), \tag{34}$$

where

$$\gamma_j = \frac{\cos\lambda_j - \cosh\lambda_j}{\sin\lambda_j - \sinh\lambda_j} \tag{35}$$

and $\lambda_1, \ldots, \lambda_{n-2}$ are the consecutive roots of the transcendental equation

$$1 - \cos\lambda \cosh\lambda = 0. \tag{36}$$

The first two assumed rigid body modes must be included in the assumed modes for completeness as they give rise to motions without flexural deformation of the beam. This type of motion could not be described by the other assumed flexural modes.

The dimensionless u is given by

$$\bar{u} = \frac{u}{L} = \sum_{i=1}^{n} \bar{q}_i(\tau)\phi_i(\xi). \tag{37}$$

The resulting dimensionless equations of motion are

$$\bar{\mathbf{H}}\ddot{\bar{\mathbf{q}}} + (\bar{\mathbf{M}} + \bar{k}_{s1}\bar{\mathbf{\Phi}}_1 + \cdots + \bar{k}_{sN}\bar{\mathbf{\Phi}}_N)\bar{\mathbf{q}} = \bar{P}\bar{\psi} \mp (\bar{k}_{n1} - \bar{k}_{s1})\bar{\mathbf{\Phi}}_1\bar{\mathbf{c}} \mp \cdots \mp (\bar{k}_{n1} - \bar{k}_{sN})\bar{\mathbf{\Phi}}_N\bar{\mathbf{c}}, \tag{38}$$

where

$$(\bar{\mathbf{H}})_{ij} = \int_0^1 \phi_i\phi_j \,\mathrm{d}\xi \tag{39}$$

$$(\bar{\mathbf{M}})_{ij} = \int_0^1 \phi_i''\phi_j'' \,\mathrm{d}\xi, \tag{40}$$

$$(\bar{\mathbf{\Phi}}_p)_{ij} = \phi_i(\bar{s}_p)\phi_j(\bar{s}_p), \tag{41}$$

and

$$(\bar{\psi})_i = \theta_i(\bar{x}) \tag{42}$$

$$(\bar{\mathbf{c}})_i = \begin{cases} \delta\phi_1 & \text{for } i = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

The matrix $\bar{\mathbf{H}}$ is equal to the identity matrix due to the orthogonality of the assumed beam functions.

3. RESULTS AND SIMULATIONS

The equation of motion has been included in numerical integration programs using a fourth-order, variable-time-step Runge–Kutta method for investigating the response of a multi-span beam subject to a stationary decaying load or a constant moving load. The deflections at the spring supports are computed at the beginning for each new time step. $\bar{k}_{sp}$ and the signs before the terms containing $(\bar{k}_{n1} - \bar{k}_{sp})$ for the spring supports are then assigned suitable values. The dimensionless deflections under the stationary load and the moving load are found to be almost identical for five-term ($n = 5$) and six-term ($n = 6$) assumed functions for $\bar{u}$. Consequently, all the following numerical results are generated using a five-term ($n = 5$) assumed function for $\bar{u}$. The beam is assumed to be straight and stationary at $\tau = 0$.

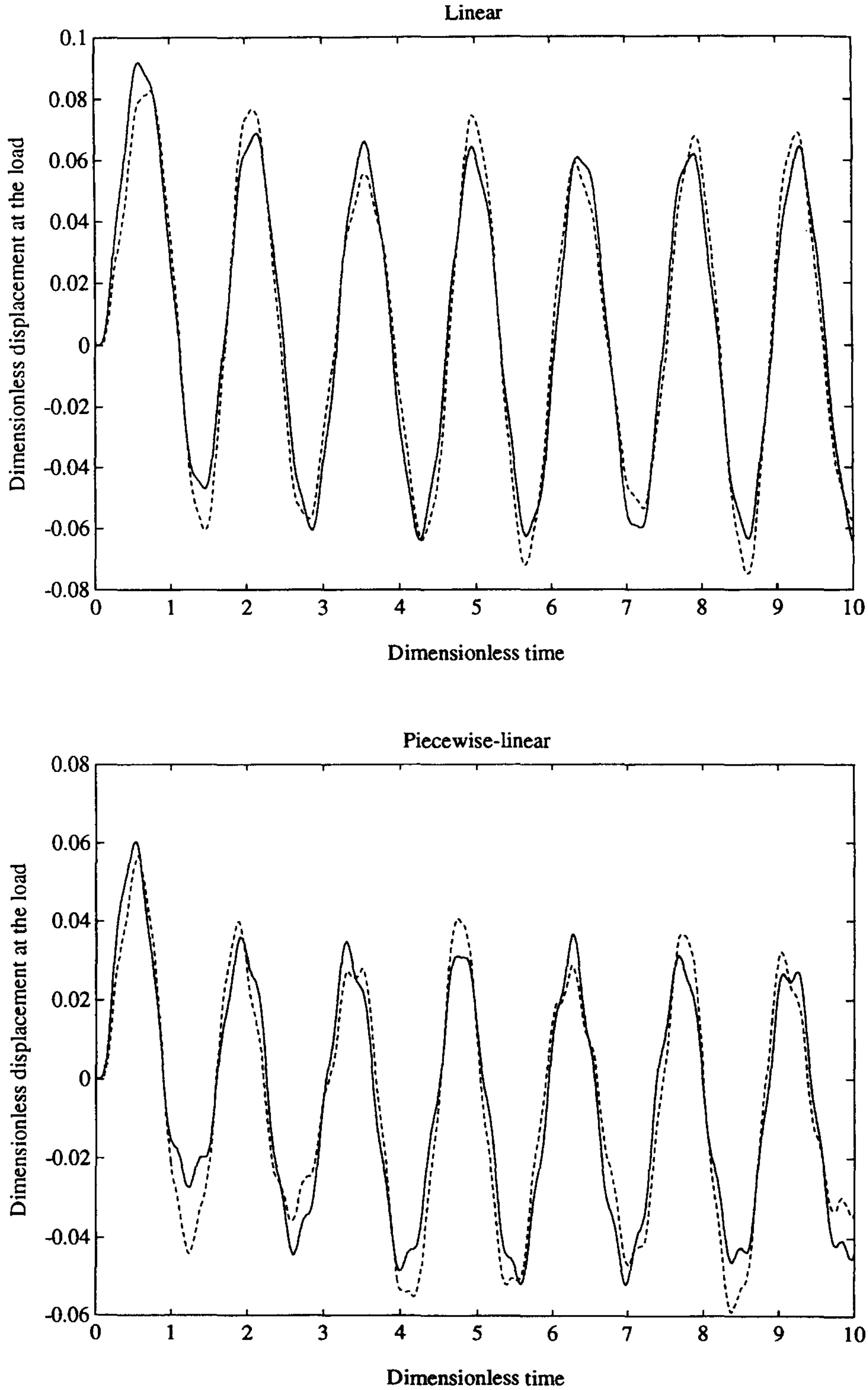

Fig. 3. Dimensionless deflection at the stationary load for a beam on two spring supports with $\bar{s}_1 = 0.1, \bar{s}_2 = 0.9, \bar{\delta} = 0.02, \bar{k}_{n1} = 10, \bar{k}_{n2} = 20, \bar{k}_{n3} = 5$, ——, $\bar{x} = 0.5$, - - -, $\bar{x} = 0.6$.

Figures 3–6 show the dimensionless deflections under a stationary decaying load with $\bar{P} = e^{-\tau}$. The transient responses for linear springs are generated with $\bar{k}_{n2}$ and $\bar{k}_{n3}$ equal to $\bar{k}_{n1}$. The magnitudes of the dimensionless deflection under the stationary load are in general smaller for piecewise-linear spring supports due to the hardening characteristic of the spring

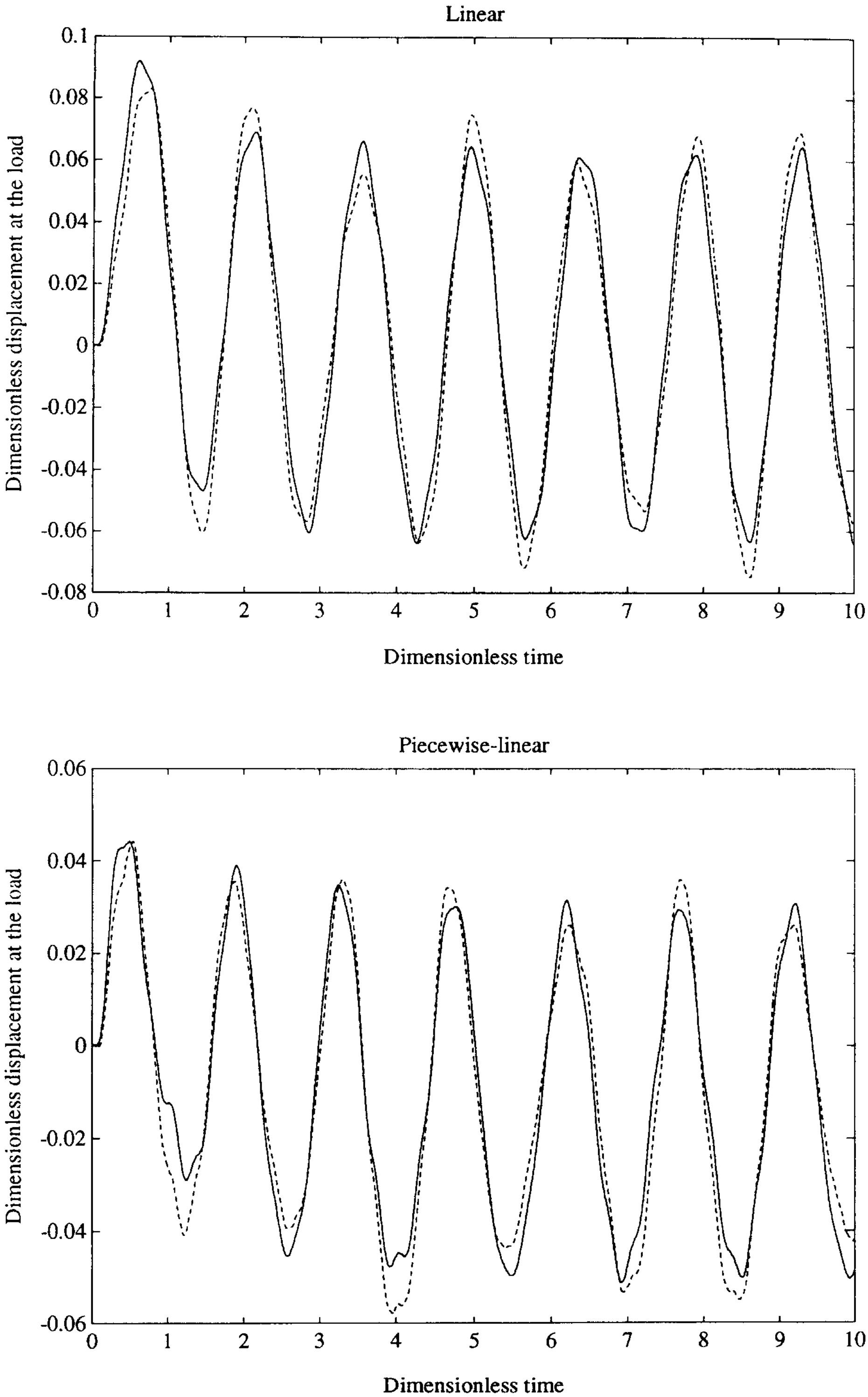

Fig. 4. Dimensionless deflection at the stationary load for a beam on two spring supports with $\bar{s}_1 = 0.1$, $\bar{s}_2 = 0.9$, $\bar{\delta} = 0.02$, $\bar{k}_{n1} = 10$, $\bar{k}_{n2} = 30$, $\bar{k}_{n3} = 3$, ——, $\bar{x} = 0.5$, - - -, $\bar{x} = 0.6$.

supports in the direction of the applied load. For a beam on two spring supports (Figs 3 and 4), the dimensionless deflection at the stationary load for the symmetric loading ($\bar{x} = 0.5$) and non-symmetric loading ($\bar{x} = 0.6$) are close to each other with no discernible trend. For a beam on five spring supports (Figs 5 and 6), the dimensionless deflection under

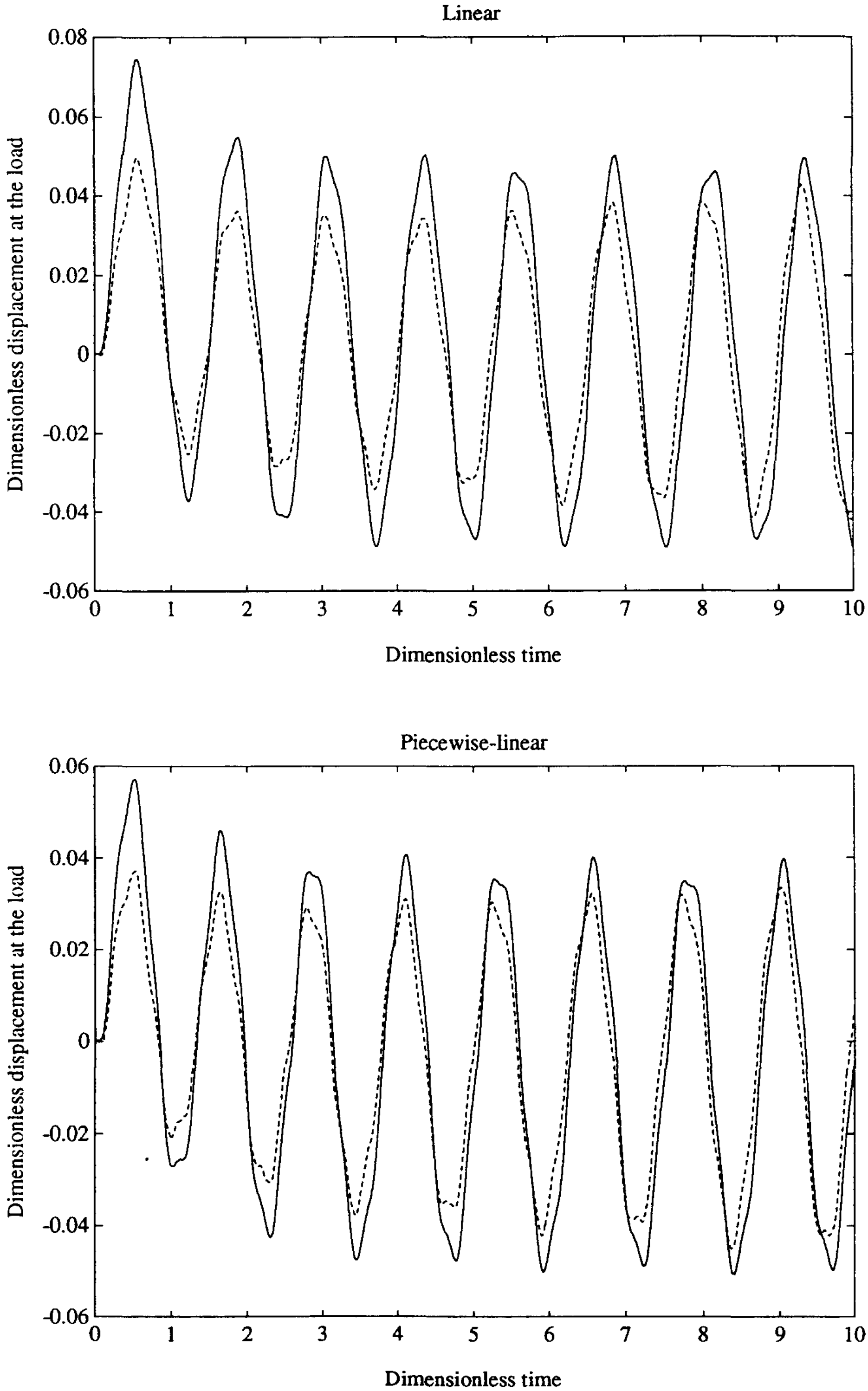

Fig. 5. Dimensionless deflection at the stationary load for a beam on five spring supports with $\bar{s}_1 = 0.1$, $\bar{s}_2 = 0.3$, $\bar{s}_3 = 0.5$, $\bar{s}_4 = 0.7$, $\bar{s}_5 = 0.9$, $\bar{\delta} = 0.01$, $\bar{k}_{n1} = 5$, $\bar{k}_{n2} = 7$, $\bar{k}_{n3} = 4$, ——, $\bar{x} = 0.5$, - - -, $\bar{x} = 0.6$.

the stationary load is in general smaller for the non-symmetric loading for both linear and piecewise–linear spring supports. Figures 4 and 6 correspond to spring supports with relatively large non-linearity. For these two cases, the responses for piecewise-linear spring supports are more complex due to the effects of higher-mode vibrations. The numerical

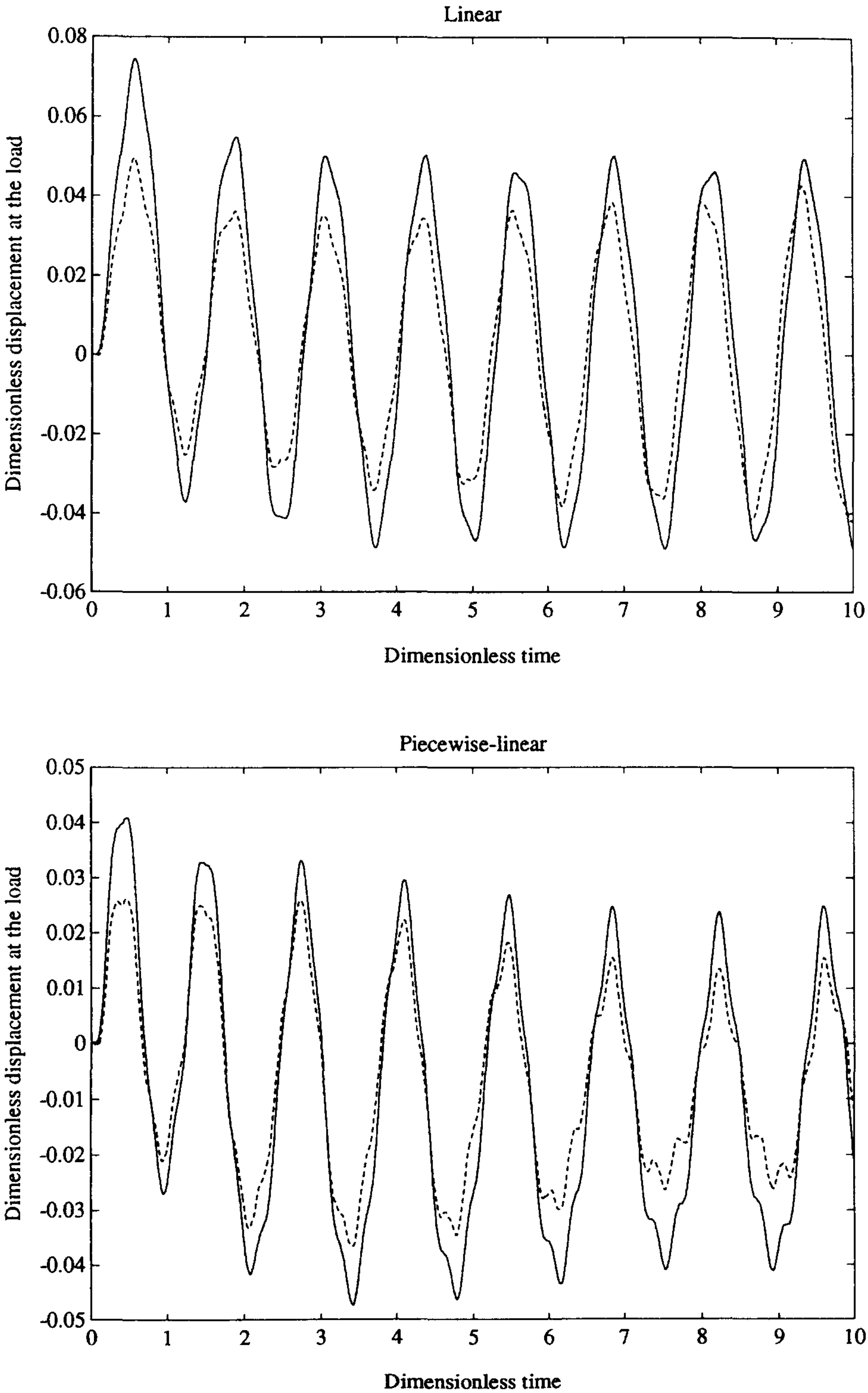

Fig. 6. Dimensionless deflection at the stationary load for a beam on five spring supports with $\bar{s}_1 = 0.1$, $\bar{s}_2 = 0.3$, $\bar{s}_3 = 0.5$, $\bar{s}_4 = 0.7$, $\bar{s}_5 = 0.9$, $\bar{\delta} = 0.01$, $\bar{k}_{n1} = 5$, $\bar{k}_{n2} = 10$, $\bar{k}_{n3} = 2$, ——, $\bar{x} = 0.5$, - - -, $\bar{x} = 0.6$.

results are in general similar to the results presented by Nagaya and Kato (1990). The initial conditions were not stated in their work and no direct comparison could be made.

The dimensionless deflections at the load for beams subject to a constant load ($\bar{P} = 1$) travelling at constant speed along the beam are shown in Figs 7 and 8. For each case, the

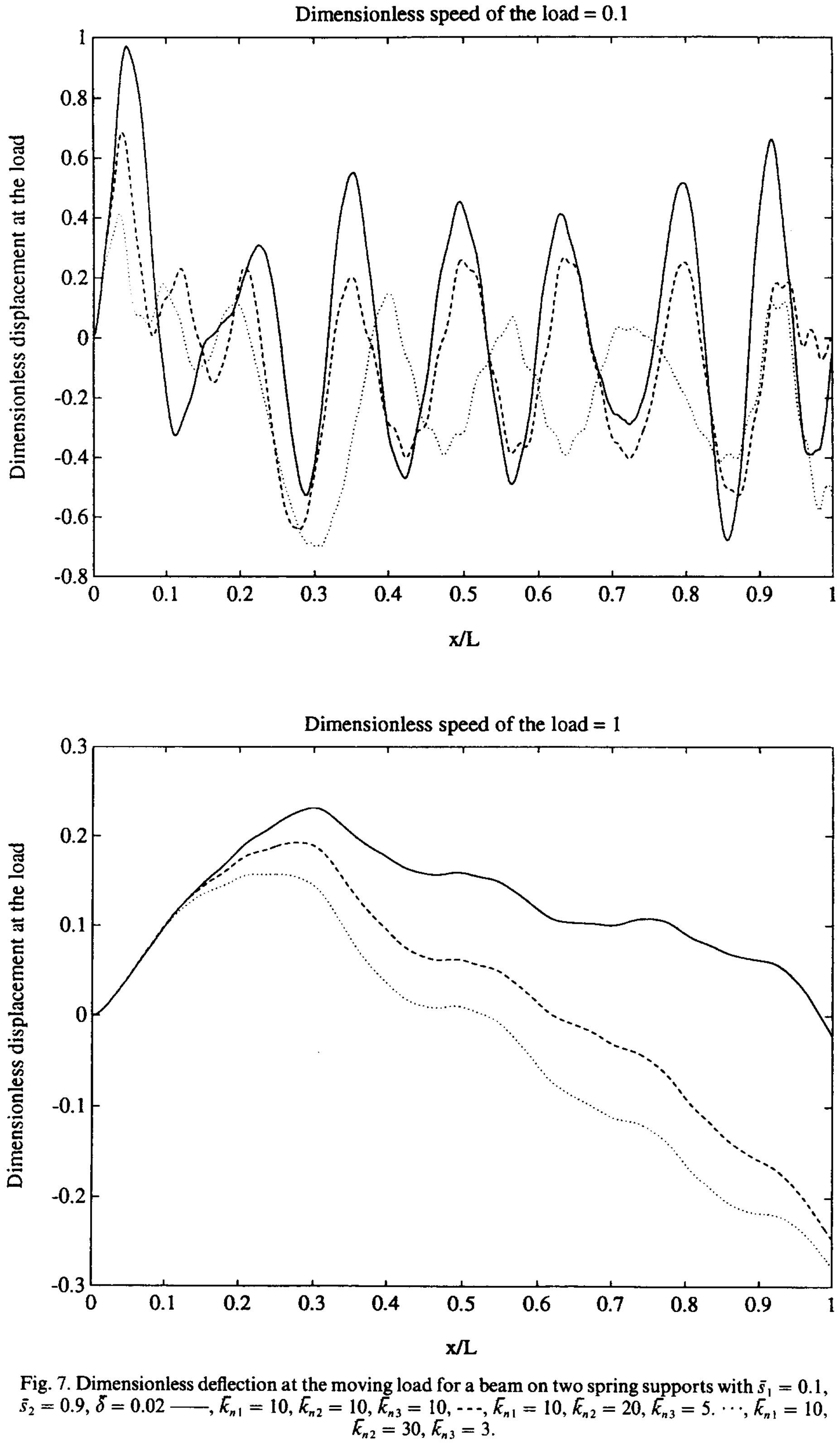

Fig. 7. Dimensionless deflection at the moving load for a beam on two spring supports with $\bar{s}_1 = 0.1$, $\bar{s}_2 = 0.9$, $\bar{\delta} = 0.02$ ——, $\bar{k}_{n1} = 10$, $\bar{k}_{n2} = 10$, $\bar{k}_{n3} = 10$, - - -, $\bar{k}_{n1} = 10$, $\bar{k}_{n2} = 20$, $\bar{k}_{n3} = 5$. · · ·, $\bar{k}_{n1} = 10$, $\bar{k}_{n2} = 30$, $\bar{k}_{n3} = 3$.

responses for a beam on linear springs, and piecewise-linear spring supports of different non-linearity are plotted on the same diagram. For a beam on two spring supports (Fig. 7), a large dimensionless speed ($\bar{x}/\tau = 1$) of the moving load is found to result in a less oscillatory transverse movement at the load for all types of spring supports. The responses

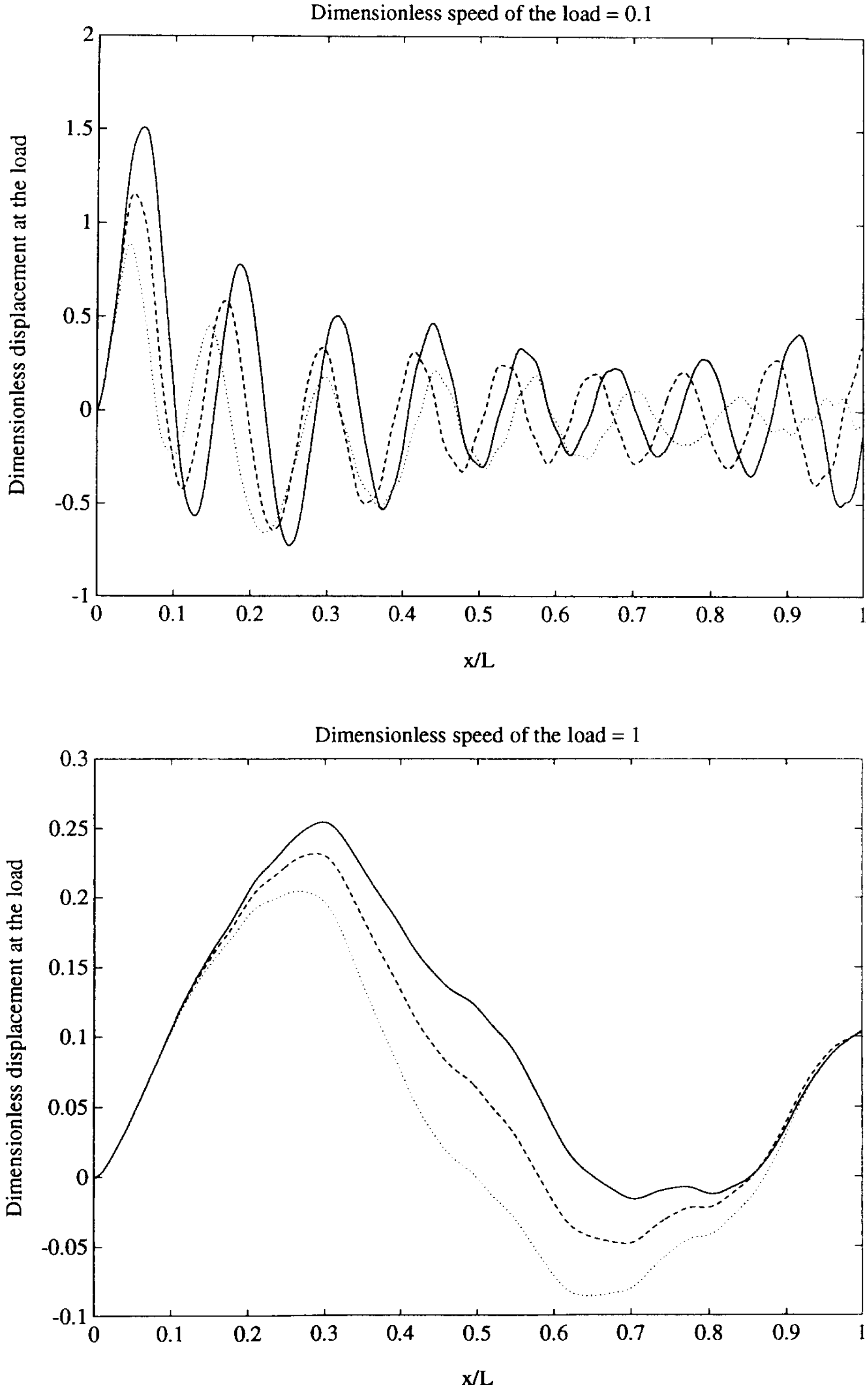

Fig. 8. Dimensionless deflection at the moving load for a beam on five spring supports with $\bar{s}_1 = 0.1$, $\bar{s}_2 = 0.3$, $\bar{s}_3 = 0.5$, $\bar{s}_4 = 0.7$, $\bar{s}_5 = 0.9$, $\bar{\delta} = 0.01$ ——, $\bar{k}_{n1} = 5$, $\bar{k}_{n2} = 5$, $\bar{k}_{n3} = 5$, - - -, $\bar{k}_{n1} = 5$, $\bar{k}_{n2} = 7$, $\bar{k}_{n3} = 4$. · · ·, $\bar{k}_{n1} = 5$, $\bar{k}_{n2} = 10$, $\bar{k}_{n3} = 2$.

for the initial period appear to be the same and diverge thereafter depending on the type of spring supports. For a slow-moving load, the responses are highly oscillatory similar to the responses of a beam under a stationary decaying load. Once again, the deflection under the moving load is generally smaller for a beam on piecewise-linear spring supports due to the hardening characteristics of the spring supports. For a beam on five spring supports (Fig. 8), the responses appear to be more oscillatory for both small and large dimensionless speeds of the load compared to the responses of a beam on two spring supports. This behavior is caused by the contribution of higher-mode vibrations due to more closely spaced spring supports.

A main advantage of the present formulation is its simplicity. The mass matrix and stiffness matrix due to the flexural rigidity of the beam in the equation of motion can be computed easily making use of the orthogonal property of the assumed beam functions. The computation of the additional matrices due to the piecewise-linear spring supports and the applied load only involves multiplications of beam functions evaluated at particular locations without numerical integration. Moreover, the complexity of the formulation does not increase with increased number of spring supports. A larger number of spring supports only increases the number of stiffness matrices due to the spring supports. These matrices are similar in form and can be computed easily.

4. CONCLUSION

The equations of motion in matrix form are formulated for the transient response of a multi-span beam on non-symmetric, piecewise-linear supports subject to a stationary load or a moving load. The non-symmetric, piecewise-linear supports are used as models for magnetic levitation supports. Results of numerical simulations are presented for beams on two and five similar spring supports subject to a stationary decaying load or a moving load. The differences between the responses of beams on linear and piecewise-linear supports have been illustrated.

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